Inverse Problem of Lagrangian Mechanics for Classically Damped Linear Multi-Degrees-of-Freedom Systems

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Lagrangians for classically damped linear multi-degrees-of-freedom dynamical systems are obtained using simple and elementary methods. Such dynamical systems are very widely used to model and analyze small amplitude vibrations in numerous naturally occurring and engineered systems. An invariant of the motion is also obtained. [DOI: 10.1115/1.4034012]

1 Introduction

There is considerable interest nowadays in the inverse problem of Lagrangian mechanics that deals with the determination of a suitable Lagrangian function, which when used with the standard apparatus of the calculus of variations, yields Euler–Lagrange equations that are identical to a given set of equations of motion. The inverse problem is of significant interest from a theoretical standpoint since the use of Lagrangians brings the descriptions of dynamical systems within the compass of the Lagrangian framework, which, along with the action principle, provides a deeper understanding of our physical world. Besides providing compact descriptions of a complex dynamical systems, they are also of considerable computational value since numerical methods using discrete Lagrangians are known to often have significant advantages over standard integration methods, for example, over Runge–Kutta algorithms.

While it is standard to use Lagrangians for undamped systems, finding Lagrangians for systems with damping is considerably more difficult. General conditions for the existence of Lagrangians were first obtained by Helmholtz and are referred to as the Helmholtz conditions [1,2]. However, even for two-degrees-of-freedom damped systems, the application of Helmohltz's conditions remains an onerous, difficult, and complex task requiring the solution of coupled partial differential equations (see, for example, Ref. [3]). Because of this, no applications of Helmholtz's conditions to systems beyond a few degrees-of-freedom appear to be available to date. A major breakthrough in this area occurred when Douglas analyzed, in considerable detail, two-degrees-of-freedom systems and obtained the necessary and sufficient conditions for the existence of Lagrangians without utilizing the Helmholtz conditions [4]. While considerable progress has been made in finding the Lagrangians for several linear and nonlinear equations of motion for damped single and two-degrees-of-freedom systems [5–8], there appears to be little work done to date that provides the results for multi-degrees-of-freedom systems of the type commonly encountered in engineering practice (see, e.g., Refs. [9–11]). Finding Lagrangians for general multi-degrees-of-freedom systems is substantially more difficult than finding them for single- or two-degrees-of-freedom systems, and this is the main reason for there being such little progress made to date in this area.

Udwadia and Cho provide some results dealing with general damped linear multi-degrees-of-freedom systems [3]. They provide Lagrangians for damped linear mechanical and structural systems with symmetric mass and stiffness matrices. The Lagrangians that they obtain for systems in which the damping matrix is skew-symmetric are physically based (also called, standard) in that the kinetic and potential energy terms in the Lagrangian can be identified. They also obtain Lagrangians for other classes of linearly damped multi-degrees-of-freedom systems in which the damping and stiffness matrices are significantly restricted to have specific structures, and in which the parameters in them depend in specific ways on elements of the mass matrix. The results in this paper go beyond those in Ref. [3] in that they are applicable to all classically damped multi-degrees-of-freedom dynamical systems.

Virtual (also called, nonstandard or unnatural) Lagrangian functions that cannot be split into kinetic and potential energy components have also gained considerable importance in recent years since the Euler–Lagrange equations that they engender can be made to match numerous nonlinear evolution equations commonly found in physics and engineering. However, here again, the focus has mainly been on systems with a very small number of degrees-of-freedom (typically one or two) [12], and the results on systems with large numbers of degrees-of-freedom, as commonly arise in engineering practice, are, to the author's knowledge, very few, if any.

This paper deals with the inverse problem of Lagrangian mechanics for classically damped linear multi-degrees-of-freedom dynamical systems, and the results obtained appear to be new. These dynamical systems are widely used to model the small amplitude motions (vibrations) of structural and mechanical components and subcomponents in the fields of civil, mechanical, and aerospace engineering, and in acoustics. Use of classically damped linear systems in analysis and design is widespread today, and quite routine, in engineered systems, like, building and bridge structures, aircraft assemblies, spacecraft structures, and automotive components. In contrast with the methods used in Ref. [3], the methods used here are simple and elementary.

The dynamical equation describing a damped linear n-degree-of-freedom system that arises most frequently in engineering applications is [9–11]

$$\tilde{M}\ddot{z} + \tilde{C}\dot{z} + \tilde{K}z = 0 \tag{1}$$

where z is a real (generalized) displacement n-vector (n by 1 vector), $\tilde{M} > 0, \tilde{K}$, and $\tilde{C}$ are real, constant, symmetric matrices, and the dots denote differentiation with respect to time, t. The matrices $\tilde{M}$, $\tilde{C}$, and $\tilde{K}$ above are the mass, damping, and stiffness matrices, respectively. Most often the system is assumed to be *classically* damped, for which the necessary and sufficient condition is $\tilde{C}M^{-1}\tilde{K} = \tilde{K}M^{-1}\tilde{C}$ [13]. This condition includes the so-called "proportional damping" situation that is commonly assumed in structural dynamics, in which the damping matrix $\tilde{C} = \alpha\tilde{M} + \beta\tilde{K}$, where α and β are constants [9–11]. Classically damped systems have so-called classical normal modes of vibration.

Making the transformation $z = \tilde{M}^{-1/2}x$ in Eq. (1) and premultiplying it by $\tilde{M}^{-1/2}$, one obtains the relation

$$\ddot{x} + C\dot{x} + Kx = 0 \tag{2}$$

where $C = \tilde{M}^{-1/2}\tilde{C}\tilde{M}^{-1/2}$ and $K = \tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2}$. The necessary and sufficient condition stated earlier for the system to be classically damped then requires that the matrices C and K commute with one another, i.e., $CK = KC$. This in turn implies that there exists an orthogonal matrix T that simultaneously diagonalizes the symmetric matrices C and K [14].

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Equation (1) is evidently equivalent to Eq. (2), and it is this equation that we shall be using in the sequel. In passive structural systems, the matrix C is positive definite.

Our aim is to first find Lagrangians whose Euler–Lagrange equations result in Eq. (2). Next, we find a quantity that this n-degree-of-freedom classically damped dynamical system conserves over time, i.e., an invariant of the motion.

2 Lagrangians for Classically Damped Linear Multi-Degrees-of-Freedom Systems

2.1 Some Simple Preliminaries. We begin by considering a single degree-of-freedom damped linear system described by the equation

$$\ddot{y} + d\dot{y} + \lambda y = 0 \tag{3}$$

where d and λ are constants. Using the transformation

$$y(t) = w(t)e^{-dt/2} \tag{4}$$

one obtains

$$\dot{y} = \dot{w}e^{-dt/2} - \frac{d}{2}we^{-dt/2} \text{ and } \ddot{y} = \ddot{w}e^{-dt/2} - d\dot{w}e^{-dt/2} + \frac{d^2}{4}we^{-dt/2} \tag{5}$$

and Eq. (3) then simplifies to

$$\ddot{w} + \left(\lambda - \frac{d^2}{4}\right)w = 0 \tag{6}$$

in which the (velocity) term in $\dot{w}$ is eliminated. A Lagrangian for Eq. (6) is then simply

$$L(w, \dot{w}) = \frac{\dot{w}^2}{2} - \frac{1}{2}\left(\lambda - \frac{d^2}{4}\right)w^2 \tag{7}$$

The first term on the right-hand side of Eq. (7) is the kinetic energy of the system described by Eq. (6), and the second member is its potential energy. The Lagrangian is the difference between the kinetic and the potential energy.

Going back to the variable $y(t)$ in Eq. (4), and noting from the first relation in Eq. (5) that $\dot{w} = (\dot{y} + (d/2)y)e^{dt/2}$, a Lagrangian of our original system given by Eq. (3) can then be written as

$$L(y, \dot{y}, t) = \frac{1}{2}\left[\left(\dot{y} + \frac{d}{2}y\right)^2 - \left(\lambda - \frac{d^2}{4}\right)y^2\right]e^{dt} \tag{8}$$

We thus have the following result.

Result 1: A Lagrangian of the equation

$$\ddot{y} + d\dot{y} + \lambda y = 0$$

is

$$L(y, \dot{y}, t) = \frac{1}{2}\left(\dot{y}^2 + d\dot{y}y + \frac{d^2}{2}y^2\right)e^{dt} - \frac{\lambda}{2}y^2e^{dt} \tag{9}$$

Proof. Equation (8) simplifies to that given above. Using the Lagrangian L in Eq. (9), it can be verified that the Euler–Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \tag{10}$$

results in Eq. (3). We note that the Lagrangian in Eq. (9) is a physically based (standard) Lagrangian, as it is deduced from Eq. (7), which is interpretable as the kinetic energy minus the potential energy. □

Remark 1. Equation (6) in the variable $w(t)$ describes an undamped system and its energy is conserved! In fact the terms related to the kinetic and potential energy of the system have been identified in Eq. (7). The energy of the system described by Eq. (6), which is conserved, is therefore given by

$$E_w = \frac{1}{2}\dot{w}^2 + \frac{1}{2}\left(\lambda - \frac{d^2}{4}\right)w^2 \tag{11}$$

which, in terms of the coordinate $y(t)$ becomes (on using Eq. (4))

$$E = \frac{1}{2}\left[\left(\dot{y} + \frac{d}{2}y\right)^2 + \left(\lambda - \frac{d^2}{4}\right)y^2\right]e^{dt} \tag{12}$$

Since E_w is constant for all time, we then find that Eq. (3) (in the variable $y(t)$ now) admits an invariant given by

$$E = \frac{1}{2}(\dot{y}^2 + d\dot{y}y + \lambda y^2)e^{dt} = \text{constant} \tag{13}$$

Result 2: Another Lagrangian for the equation

$$\ddot{y} + d\dot{y} + \lambda y = 0$$

is

$$L(y, \dot{y}, t) = \frac{1}{2}\dot{y}^2e^{dt} - \frac{\lambda}{2}y^2e^{dt} \tag{14}$$

Proof. Substitution of this Lagrangian in Eq. (10) gives Eq. (3). □

2.2 Lagrangians for Classically Damped Linear Multi-Degrees-of-Freedom Systems. For the dynamical system described by Eq. (2) to be classically damped we must have $CK = KC$. As mentioned before, when these matrices commute, there exists an orthogonal matrix T (i.e., $T^TT = I$) such that [14]

$$CT = TD \text{ and } KT = T\Lambda \tag{15}$$

where the diagonal matrix

$$D = \text{diag}(d_1, d_2, \ldots, d_n) \tag{16}$$

and the diagonal matrix

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{17}$$

Using the change of variable

$$x(t) = Ty(t) \tag{18}$$

in Eq. (2), where we denote the n-vectors $x(t) = (x_1, x_2, \ldots, x_n)^T$ and $y(t) = (y_1, y_2, \ldots, y_n)^T$, we get

$$T\ddot{y} + CT\dot{y} + KTy = 0 \tag{19}$$

which upon premultiplication by T^T gives, by virtue of relations (15), the n uncoupled set of modal equations

$$\ddot{y}_i + d_i\dot{y} + \lambda_i y = 0, \quad i = 1, 2, \ldots, n \tag{20}$$

It should be noted that though in structural dynamics the matrices C and K are usually taken to be positive definite, this is not required for uncoupling of the equation of motion given in Eq. (2) to yield Eq. (20) via the transformation (18) as has been done here; only symmetry of the matrices C and K is needed.

Each of the n equations in equation set (20) represents a linearly damped system of the type discussed in Sec. 2.1, and so Lagrangians for each of these n uncoupled equations are known from Results 1 and 2 in Sec. 2.1.

Starting with Result 1, a Lagrangian for the system of equations described by the equation set (20) can then be simply written as

$$L(y,\dot{y},t)=\sum_{i=1}^{n}\left[\frac{1}{2}\left(\dot{y}_i^2+d_i\dot{y}_iy_i+\frac{d_i^2}{2}y_i^2\right)e^{d_i t}-\frac{\lambda_i}{2}y_i^2e^{d_i t}\right] \tag{21}$$

The ith term in the summation above is a Lagrangian for the ith equation of the set given in Eq. (20). Furthermore, each Lagrangian has been shown to have a physical meaning in terms of the kinetic and potential energy.

As with the single degree-of-freedom system, all that remains now is to transform Eq. (21) back to the original variable, $x(t)$.

We begin by noticing that Eq. (21) can be written in a more compact form as

$$L(y,\dot{y},t)=\left[\frac{1}{2}\left(\dot{y}^Te^{Dt}\dot{y}+\dot{y}^Te^{Dt}Dy+\frac{1}{2}y^Te^{Dt}D^2y\right)-\frac{1}{2}y^Te^{Dt}\Lambda y\right] \tag{22}$$

Note that D is a diagonal matrix (see Eq. (16)) and hence e^{Dt} is also diagonal; furthermore, $e^{Dt}D^m=D^me^{Dt}$ for any integer m. Also, $\Lambda e^{Dt}=e^{Dt}\Lambda$, since both the matrices on either side of the equality are diagonal.

To transform the Lagrangian in Eq. (22) back to our original variable $x(t)$, we use the relation $y(t)=T^Tx(t)$, which follows from Eq. (18). This gives

$$L(x,\dot{x},t)=\left[\frac{1}{2}\left(\dot{x}^TTe^{Dt}T^T\dot{x}+\dot{x}^TTe^{Dt}DT^Tx+\frac{1}{2}x^TTe^{Dt}D^2T^Tx\right)-\frac{1}{2}x^TTe^{Dt}\Lambda T^Tx\right] \tag{23}$$

Equation (15) can be expressed as $C=TDT^T$, so that $Te^{Dt}T^T=e^{Ct}$. Furthermore, since T is an orthogonal matrix, $Te^{Dt}DT^T=Te^{Dt}T^TTDT^T=e^{Ct}C$. Also, $Te^{Dt}D^2T^T=Te^{Dt}T^TTD^2T^T=e^{Ct}C^2$, and similarly $Te^{Dt}\Lambda T^T=e^{Ct}K$.

Using these relations in Eq. (23), we then have our first main result.

Result 3: A Lagrangian for the classically damped n-degree-of-freedom system

$$\ddot{x}+C\dot{x}+Kx=0$$

is given by

$$L(x,\dot{x},t)=\frac{1}{2}\left(\dot{x}^Te^{Ct}\dot{x}+\dot{x}^Te^{Ct}Cx+\frac{1}{2}x^Te^{Ct}C^2x\right)-\frac{1}{2}x^Te^{Ct}Kx \tag{24}$$

Proof. We obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right)&=\frac{d}{dt}\left(e^{Ct}\dot{x}+\frac{1}{2}e^{Ct}Cx\right)\\&=e^{Ct}\ddot{x}+e^{Ct}C\dot{x}+\frac{1}{2}e^{Ct}C\dot{x}+\frac{1}{2}e^{Ct}C^2x\\&=e^{Ct}\ddot{x}+\frac{3}{2}e^{Ct}C\dot{x}+\frac{1}{2}e^{Ct}C^2x\end{aligned} \tag{25}$$

and noting that $KC=CK$ we get

$$\frac{\partial L}{\partial x}=\frac{1}{2}e^{Ct}C\dot{x}+\frac{1}{2}e^{Ct}C^2x-e^{Ct}Kx \tag{26}$$

Using these relations in the Euler–Lagrange equation the result follows. □

One could also start with the expression for the Lagrangian given in Eq. (14) (see Result 2 in Sec. 2.1) for each of the n uncoupled equations of the set (20). A Lagrangian for the system of equation (20) can be written, in a fashion similar to what was done above, as

$$L(y,\dot{y},t)=\frac{1}{2}\dot{y}^Te^{Dt}\dot{y}-\frac{1}{2}y^Te^{Dt}\Lambda y \tag{27}$$

Transforming back to the coordinates $x(t)=Ty(t)$ then yields

$$L(x,\dot{x},t)=\frac{1}{2}\dot{x}^TTe^{Dt}T^T\dot{x}-\frac{1}{2}x^TTe^{Dt}\Lambda T^Tx$$

which gives the following result.

Result 4: A Lagrangian for the classically damped n-degree-of-freedom system

$$\ddot{x}+C\dot{x}+Kx=0$$

is given by

$$L(x,\dot{x},t)=\left[\frac{1}{2}\dot{x}^Te^{Ct}\dot{x}-\frac{1}{2}x^Te^{Ct}Kx\right] \tag{28}$$

Proof: Noting that the matrices C and K commute, substitution of Eq. (28) in the Euler–Lagrange equation verifies the result. □

A Lagrangian is arguably the most compact way of describing the information contained in a dynamical system. This information can be "unpacked" by simply using the calculus of variations (also called the action principle) to generate the Euler–Lagrange equations that govern its dynamical evolution. Besides yielding the evolution equations, Lagrangians help decipher symmetries, and Lagrangian formulations are useful in studying stability, in applying perturbation methods, and in finding invariants. We next find an invariant for our classically damped n-degree-of-freedom dynamical system.

2.3 An Invariant for Classically Damped Linear Multi-Degree-of-Freedom Systems.

The Lagrangian of the classically damped linear system described in Result 3 is time dependent and so the Jacobi integral cannot be directly used to find a conserved quantity [15]. However, Remark 1 shows that each of the n uncoupled equations in Eq. (20) conserves a scalar quantity. Using Eq. (13) given in Remark 1, the system of equations described by the equation set (20) then conserves the quantity

$$E=\sum_{i=1}^{n}E_i=\sum_{i=1}^{n}\frac{1}{2}\left[\dot{y}_i^2+d_i\dot{y}_iy_i+\lambda_iy_i^2\right]e^{d_i t} \tag{29}$$

which can be more compactly written as

$$E=\frac{1}{2}\left(\dot{y}^Te^{Dt}\dot{y}^T+\dot{y}^Te^{Dt}Dy+y^Te^{Dt}\Lambda y\right) \tag{30}$$

Rewriting this in terms of the variable $x(t)$ and using Eq. (15) one obtains

$$E=\frac{1}{2}(\dot{x}^TTe^{Dt}T^T\dot{x}+\dot{x}^TTe^{Dt}DT^Tx+x^TTe^{Dt}\Lambda T^Tx) \tag{31}$$

which on simplification gives the following result.

Result 5: The classically damped n-degree-of-freedom system

$$\ddot{x}+C\dot{x}+Kx=0$$

conserves the quantity

$$E=\frac{1}{2}(\dot{x}^Te^{Ct}\dot{x}+\dot{x}^Te^{Ct}Cx+x^Te^{Ct}Kx) \tag{32}$$

The quantity E is thus an invariant of the motion of the dynamical system.

Proof: Noting Eq. (15) and the fact that $CK = KC$ the following simplifications result:

$$Te^{Dt}T^T = e^{Ct} \tag{33}$$

$$Te^{Dt}DT^T = Te^{Dt}T^TTDT^T = e^{Ct}C \tag{34}$$

$$Te^{Dt}\Lambda T^T = Te^{Dt}T^TT\Lambda T^T = e^{Ct}K = Ke^{Ct} \tag{35}$$

Using them in Eq. (31), the result follows. Though a bit more time consuming, it can also be directly shown using Eq. (32) that $(dE/dt) = 0$ along the trajectories of the system. □

We next explore the invariant of motion where we use the Lagrangian given in Result 2 instead of using Remark 1. Does an invariant different from that given in Eq. (32) result?

As before, that Lagrangian in Eq. (14) of Result 2 is an explicit function of time, and therefore, one cannot directly avail of the Jacobi integral to provide an invariant of motion [16]. However, using the transformation $y(t) = w(t)e^{-dt/2}$ in Eq. (14) gives

$$L(w, \dot{w}) = \frac{1}{2}\left(\dot{w}^2 - dw\dot{w} + \frac{d^2w^2}{4}\right) - \frac{\lambda w^2}{2} \tag{36}$$

which is now no longer an explicit function of t. One can now use the Jacobi integral to find an invariant. The Jacobi integral for the Lagrangian given in Eq. (36) is given by [16]

$$\begin{aligned} J &= \dot{w}\frac{\partial L}{\partial \dot{w}} - L(w, \dot{w}) = \dot{w}\left(\dot{w} - \frac{dw}{2}\right) - L(w, \dot{w}) \\ &= \frac{1}{2}\left(\dot{w}^2 - \frac{1}{4}d^2w^2 + \lambda w^2\right) \end{aligned} \tag{37}$$

Transforming the expression above back to the coordinate $y(t)$ by substituting $w(t) = y(t)e^{dt/2}$ the invariant J becomes

$$J = \frac{1}{2}\left(\dot{y}^2 + dy\dot{y} + \lambda y^2\right)e^{dt} \tag{38}$$

which, as shown in Remark 1, is identical to the conserved quantity E obtained therein by using the Lagrangian in Result 1!

Continuing the argument, as before (see Eqs. (29) and (30)), in order to find the invariant for an n-degree-of-freedom classically damped system, we observe that the Lagrangians given in Eqs. (8) and (14) result in the same invariant of motion that is given in Eq. (32) (see Result 5).

3 Conclusions

This paper obtains Lagrangians for multi-degree-of-freedom classically damped linear systems through the use of very elementary methods. An invariant for these damped multi-degree-of-freedom dynamical systems is also found. These results appear to be new.

Classically damped linear dynamical systems are widely used in numerous fields of science and engineering to model, analyze, and design, physical systems that undergo small amplitude vibrations. Besides providing a very concise description of a dynamical system, Lagrangians, provide, arguably, one of the best platforms for a deeper understanding of the physics underlying the dynamical behavior of numerous natural and engineered systems.

It appears fortuitous that Lagrangians and invariants can be found with such elementary mathematical machinery for this class of damped multi-degree-of-freedom dynamical systems that enjoy such widespread use.

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